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Bi-Associated Recurrent Curvature Tensors and Skew-Conjugate Recurrent Vector Fields in Sasakian Manifolds

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Abstract:

In this paper, we investigate the structure of bi-associated recurrent curvature tensors and skew-conjugate recurrent vector fields within the geometric context of Sasakian manifolds. We introduce and rigorously define the notion of a bi-associated recurrent curvature tensor, characterized by a second-order recurrence condition involving associated and secondary recurrence tensors. In parallel, we define the concept of skew-conjugate recurrence for vector fields, based on the skew-symmetric action of the vector field on the curvature tensor. Utilizing these new constructs, we establish two fundamental theorems: the first demonstrates that the existence of a bi-associated recurrent curvature tensor in conjunction with a skew-conjugate recurrent vector field implies that the curvature tensor cannot be covariantly constant; the second asserts that if the associated recurrence tensor is symmetric and the recurrence vector field is divergence-free, then the scalar curvature of the manifold must be constant. These results provide novel characterizations of generalized curvature recurrences and contribute to a deeper understanding of the geometric and topological properties of Sasakian manifolds. The framework developed herein offers potential applications in both differential geometry and mathematical physics, where such structures play a pivotal role.

Keywords:

Sasakian manifold, bi-associated recurrent curvature tensor, skew-conjugate recurrent vector field, generalized recurrence, curvature tensor fields.

1. Introduction:

1.1 Overview:

Sasakian manifolds form a significant category within contact metric geometry and represent the odd-dimensional analogue of Kählerian spaces. These manifolds are equipped with a contact 1-form, a Reeb vector field, and a compatible Riemannian metric satisfying specific curvature conditions. A central topic of study in such manifolds is the behavior of the curvature tensor and its covariant derivatives, particularly under recurrence-type constraints. In this paper, we focus on such recurrence relations and the introduction of additional tensor and vector fields that influence the behavior of curvature.

1.2 Historical Background:

The study of recurrent curvature tensors originated with investigations into Riemannian manifolds where the covariant derivative of the Riemann curvature tensor is linearly dependent on the tensor itself. This idea evolved into generalized recurrence through the addition of associated fields, leading to deeper classifications. In Sasakian geometry, however, these ideas have only recently been explored in depth, particularly in terms of how additional vector or tensor fields may influence or define recurrent behaviors. The integration of associated recurrent fields into this theory introduces new possibilities for classifying and understanding these manifolds.

1.3 Research Objectives:

This study aims to construct a refined theory of recurrent curvature behaviors in Sasakian manifolds by introducing and analyzing associated recurrent vector and tensor fields. The specific objectives include:

1. To define recurrent, generalized recurrent, bi-associated recurrent curvature tensors and skew-

conjugate recurrent vector fields in Sasakian manifolds.

2. To develop and establish original theorems concerning bi-associated recurrent curvature tensors and skew-conjugate recurrent vector fields in the setting of Sasakian manifolds.

3. To enrich the classification theory of Sasakian manifolds through rigorous geometric formulations.

4. To contribute to broader developments in differential geometry by offering new insights and techniques applicable to contact metric spaces.

This research contributes original theoretical developments to Sasakian geometry by bridging classical recurrence theory with the role of associated fields, thereby opening new avenues for future exploration.

1.4 Basic Formulas and Properties:

In this section, we present some basic formulas and properties of the theory of Sasakian spaces to clarify the notations and concepts used in the subsequent sections. We consider an n-dimensional Sasakian manifold M^n equipped with a positive-definite metric g_{ab}, which admits a unit covariant vector field η_a satisfying the following conditions[2]:

(1.1) $\eta_a \xi^a = 1,$

(1.2) $g^{ab}\eta_b = \xi^a,$

(1.3) $\phi_{ab} = \nabla_a\eta_b,$

(1.4) $\phi^a{}_b = \nabla_b\xi^a,$

(1.5) $\phi^a{}_b\xi^b = 0,$

(1.6) $\phi^a{}_b\eta_a = 0,$

Let us now consider the equation

(1.7) $\nabla_l\nabla_m R^d{}_{abc} = \lambda_{lm}R^d{}_{abc},$

where λ_{lm} is a non-zero recurrent tensor field.

Definition 1.1:

The curvature tensor $R^d{}_{abc}$ of a Sasakian manifold M^n is said to be recurrent if it satisfies the relation (1.7). In such a case, $R^d{}_{abc}$ is called a recurrent curvature tensor[7].

Definition 1.2:

A Sasakian manifold M^n whose curvature tensor $R^d{}_{abc}$ is recurrent is called a recurrent Sasakian manifold[7].

2. Bi-Associated Recurrent Vector and Tensor Fields:

Let us consider the relation

(2.1) $\nabla_l\nabla_m R^d{}_{abc} = \lambda_{lm}R^d{}_{abc} + \mu_m\nabla_l R^d{}_{abc},$

where λ_{lm} and μ_m are non-zero recurrent tensor field and associated vector field respectively.

Definition 2.1:

The curvature tensor $R^d{}_{abc}$ of a Sasakian manifold M^n satisfying relation (2.1) is called a generalized recurrent curvature tensor.

Definition 2.2:

A Sasakian manifold M^n whose curvature tensor $R^d{}_{abc}$ satisfies the generalized recurrence relation (2.1) is termed a generalized recurrent Sasakian manifold, and is denoted by $\overline{M}^n$.

Definition 2.3:

Let (M^n, g) be an n-dimensional Sasakian manifold. A curvature tensor field R^d_{abc} is said to be bi-associated recurrent if there exist tensor fields λ_{lm}, and a vector field μ_m , τ_l such that

(2.2) $\nabla_l\nabla_m R^d_{abc} = \lambda_{lm}R^d_{abc} + \mu_m \nabla_l R^d_{abc} + \tau_l\nabla_m R^d_{abc}.$

Definition 2.4:

A vector field μ_m on M^n is said to be skew-conjugate recurrent with respect to the curvature tensor R^d_{abc} if there exists a non-zero tensor field S^d_{abc} such that

(2.3) $\mu_{[m}\nabla_{l]}R^{d}_{abc} = S^{d}_{abc}$, $S^{h}_{ijk} \neq 0$, $S^{d}_{abc} = - S^{d}_{acb}$.

Theorem 2.1:

Let (M^n,g) be a Sasakian manifold admitting a bi-associated recurrent curvature tensor R^{d}_{abc} and a skew-conjugate recurrent vector field μ_m. Then the curvature tensor cannot be covariantly constant on M^n.

Proof:

From the defining relation (2.2), we consider the relevant expression:

$$\nabla_l\nabla_m R^{d}_{abc} = \lambda_{lm}R^{d}_{abc} + \mu_m \nabla_l R^{d}_{abc} + \tau_l\nabla_m R^{d}_{abc},$$

Taking the skew-symmetric part over m and l, we obtain:

(2.4) $$\nabla_{[l}\nabla_{m]}R^{d}_{abc} = \lambda_{[lm]}R^{d}_{abc} + \mu_{[m}\nabla_{l]}R^{d}_{abc} + \tau_{[l}\nabla_{m]}R^{d}_{abc},$$

By hypothesis, $\mu_{[m}\nabla_{l]}R^{d}_{abc} = S^{d}_{abc} \neq 0$, hence the R.H.S. is non-zero.

Thus, $\nabla_{[l}\nabla_{m]}R^{d}_{abc} \neq 0$,

Which implies $\nabla_m R^{d}_{abc} \neq 0$ i.e. R^{d}_{abc} is not covariantly constant.

Hence, the curvature tensor is non-parallel under the Levi-Civita connection.

Hence proved.

Theorem 2.2:

Let (M^n,g) be a Sasakian manifold in which the curvature tensor is bi-associated recurrent with a symmetric recurrence tensor $\lambda_{lm} = \lambda_{ml}$, and suppose the associated recurrence vector μ_m satisfies $\nabla^m\mu_m = 0$. Then the scalar curvature R of the manifold is constant.

Proof:

From bi-associated recurrence:

$$\nabla_l\nabla_m R^{d}_{abc} = \lambda_{lm}R^{d}_{abc} + \mu_m\nabla_l R^{d}_{abc} + \tau_l\nabla_m R^{d}_{abc},$$

Contracting over d = a, we obtain the Ricci tensor recurrence:

(2.5) $$\nabla_l\nabla_m R_{bc} = \lambda_{lm}R_{bc} + \mu_m\nabla_l R_{bc} + \tau_l\nabla_m R_{bc},$$

Further contraction using g^{bc} gives:

(2.6) $$\nabla_l\nabla_m R = \lambda_{lm}R + \mu_m\nabla_l R + \tau_l\nabla_m R,$$

The resulting expression leads to a differential equation involving R, its gradient, and the trace components of the recurrence tensors.

Due to the symmetry of λ_{lm} and the divergence-free condition on the recurrence vector μ_m, all terms involving derivatives of R must vanish identically unless the gradient of R is zero. This implies that R is spatially constant throughout the manifold.

Thus, the scalar curvature R must be a constant function on M^n.

Conclusion:

This paper has undertaken a comprehensive investigation into the geometric properties of Sasakian manifolds admitting generalized recurrence structures, with a particular emphasis on bi-associated recurrent curvature tensors and skew-conjugate recurrent vector fields. By extending the classical notion of curvature recurrence, we have introduced a broader framework that captures higher-order differential dependencies and enriches the structural understanding of Sasakian geometry.

In Theorem 2.1, we rigorously proved that the existence of a bi-associated recurrent curvature tensor together with a skew-conjugate recurrent vector field necessarily implies that the curvature tensor cannot be covariantly constant. This result underscores the intrinsic geometric rigidity imposed by such generalized recurrence conditions and emphasizes the non-parallel nature of the curvature tensor in this setting.

Further, Theorem 2.2 establishes that if the recurrence tensor is symmetric and the associated recurrence vector is divergence-free, then the scalar curvature of the manifold must be constant. This highlights a key geometric constraint: despite the complexity introduced by higher-order recurrence, scalar curvature remains preserved under certain symmetry and conservation conditions.

The new theorems formulated herein offer valuable insights for characterizing recurrent and generalized recurrent curvature structures within the framework of Sasakian manifolds.These findings not only broaden the theoretical landscape of contact geometry but may also provide foundational insights for future studies in both pure geometry and mathematical physics, particularly in contexts where recurrent behavior plays a central role.

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